Hamiltonicity of The Graph *G(n, k)* of The Johnson Scheme

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Abstrak

Grafik G(n,k) of the Johnson Scheme, biasa disebut juga "Slice of the Cube", adalah grafik tanpa arah (undirected graph) dimana setiap vertex adalah k-subset dari n-set. Dua vertex A dan B adalah berdekatan dalam grafik G jika dan hanya jika $| A \cap B | = k - 1$.
Grafik ini telah dipelajari secara intensif dan beberapa ciri telah ditemukan seperti hamiltonicity, diameter, connectivity dan wide-diameter dari grafik tersebut.
Dalam tulisan ini akan ditunjukkan alternatif pembuktian dari salah satu ciri, yaitu hamiltonicity dari grafik G(n,k).

Kata kunci*: graph G(n,k) of the Johnson Scheme, subgraph, hamiltonian path, hamiltonian cycle, hamiltonicity.*

1. Introduction

There are two classes of interconnection network topologies, *static* and *dynamic*. In a static network, the network nodes are interconnected by point-to-point links in some fixed topology. Sometimes, these networks are also called *direct* networks because the interconnecting links are directly connected to the network nodes. Static interconnection network topologies are derived from graphs in which nodes represent processors and edges represent the dedicated links among processors. In multi computer interconnection, static networks are used frequently because they have an important advantage. The degree of a node either remains fixed regardless of the size of the network or grows very slowly with network size. This allows very large networks to be constructed. Some examples of the static interconnection networks used in commercial and experimental systems are as follows. Hypercube is used in Connection Machine CM-2, 2-D Mesh is used in Stanford Dash, Intel Paragon, Intern Touchstone Delta and Illinois Illiac IV, 3-D Mesh is used in MIT J-Machine.

The fact that many of the static interconnection networks used in commercial and experimental systems gives us the motivation to study the graph *G(n, k)* of the Johnson Scheme which as a possible of the static interconnection network topology. Moreover, by studying this graph, we hope to contribute some information regarding the efficiency and effectiveness of this graph as an interconnection network.

2. Definition of the graph *G(n, k)* of the Johnson Scheme

The graph $G(n, k)$ of the Johnson Scheme is the undirected graph where the vertices are all the k-subsets of a fixed n-set. Two vertices A and B are adjacent if and only if $| A \cap B | = k-1$.

For example:
Let $n = 6$, $k = 3$, V is a set of vertices, and we denote the elements of a vertex with numbers, then we shall get.

$n = \{1,2,3,4,5,6\}$
$V = \{\{1,2,3\}, \{1,2,4\}, \{1,2,5\}, \{1,2,6\}, \{1,3,4\}, \{1,3,5\}, \{1,3,6\}, \{1,4,5\}, \{1,4,6\}, \{1,5,6\}, \{2,3,4\}, \{2,3,5\}, \{2,3,6\}, \{2,4,5\}, \{2,4,6\}, \{2,5,6\}, \{3,4,5\}, \{3,4,6\}, \{3,5,6\}, \{4,5,6\}\}$

We shall assume that if $\{a_1, a_2, a_3, \ldots, a_n\}$ denotes the elements of a vertex, then $a_1 < a_2 < a_3 < \ldots < a_n$.

From the definition of the graph $G(n, k)$ of the Johnson Scheme, we know that the total number of vertices is $\binom{n}{k}$. Each vertex will have $nk - k^2$ edges incident to it, or in other words, the graph is $nk - k^2$ regular.

Consider the example. The total number of vertices is 20 and each vertex has 9 edges incident to it.

Consider the order of the elements in a vertex, we make a consensus that the order of the elements in a vertex is in non-decreasing order. This means that vertex with elements {1,3,2} or {2,1,3} or {2,3,1} or {3,1,2} or {3,2,1} will be denoted by {1,2,3}.

By this consensus, we divide the graph $G(n, k)$ of the Johnson Scheme into subgraphs S_i such that each subgraph S_i has vertices with a common first element, a_1.

Consider the example. The graph will have 4 subgraphs.

$S_1 = \{\{1,2,3\}, \{1,2,4\}, \{1,2,5\}, \{1,2,6\}, \{1,3,4\}, \{1,3,5\}, \{1,3,6\}, \{1,4,5\}, \{1,4,6\}, \{1,5,6\}\}$
$S_2 = \{\{2,3,4\}, \{2,3,5\}, \{2,3,6\}, \{2,4,5\}, \{2,4,6\}, \{2,5,6\}\}$
$S_3 = \{\{3,4,5\}, \{3,4,6\}, \{3,5,6\}\}$
$S_4 = \{\{4,5,6\}\}$

In general, the graph $G(n, k)$ of the Johnson Scheme will have $n - k + 1$ such subgraphs, since the first element in the last subgraph is $(n - k) + 1$. *Figure 1* is a representation of the graph $G(6, 3)$ of the Johnson Scheme.

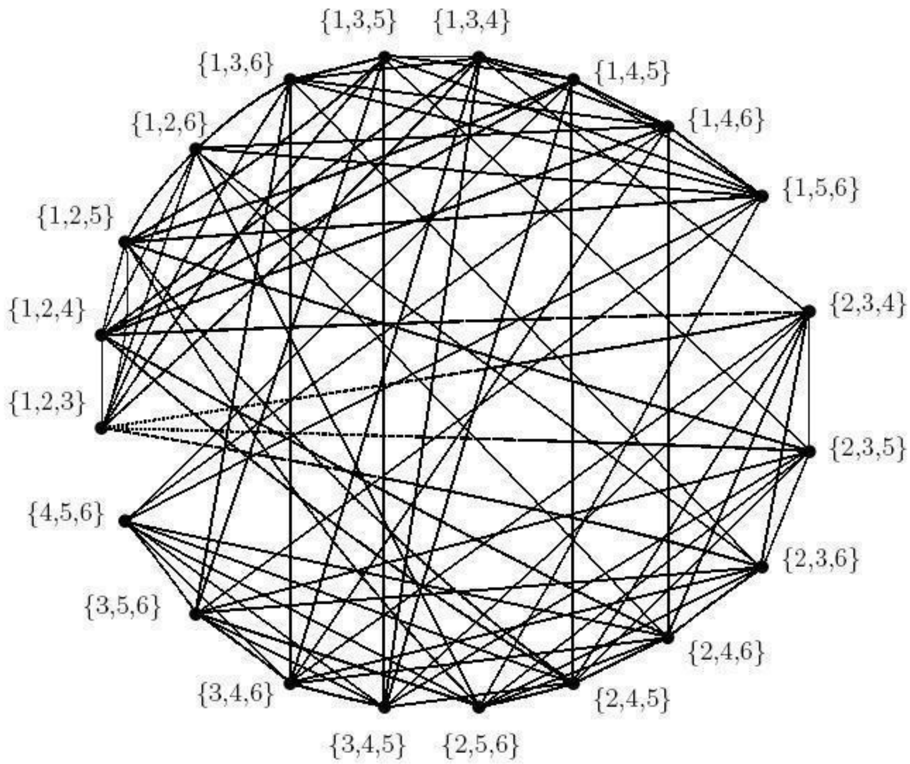

Figure 1 Graph G(6,3) of the Johnson Scheme

3. Hamiltonicity

In this paper we shall determine one of many properties of graph G(*n, k*) of the Johnson Scheme which is whether the graph is Hamiltonian or not. Even though this property has been proven by Zhang, F. *et al.*, 1997, we shall prove it in another way.

Definition

A hamiltonian path of graph is a path which passes once through each vertex of a graph (except that it may be closed, that is, its first and last vertices may be the same).

A closed hamiltonian path or a hamiltonian cycle is a hamiltonian path where the first and last vertices are the same. A graph possessing such a circuit is called hamiltonian.

Hamiltonian Graph

We first prove the following lemmas.

Lemma 1: *Each subgraph S_i of the graph $G(n, k)$ of the Johnson Scheme has a hamiltonian path.*

Proof:

Consider subgraph S_i of $G(n, k)$. For any $A \in V$ where V is the vertex set of $G(n, k)$, assume that $A = \{a_1, a_2, a_3, \ldots, a_k\}$, where a_i is ith element in n, for $i = 1, 2, 3, \ldots, n$. We also assume without loss of generality that $a_1 < a_2 < a_3 < \ldots < a_n$. Note that every vertex has exactly k elements.

Thus, each subgraph S_i has a general form as follows.

$$
\begin{aligned}
S_i = \{ & \{a_i, a_{(i+1)}, a_{(i+2)}, \ldots, a_{(i+k-2)}, a_{(i+k-1)}\}, \\
& \{a_i, a_{(i+1)}, a_{(i+2)}, \ldots, a_{(i+k-2)}, a_{(i+k)}\}, \\
& \ldots, \\
& \{a_i, a_{(i+1)}, a_{(i+2)}, \ldots, a_{(i+k-2)}, a_n\}, \\
& \{a_i, a_{(i+1)}, a_{(i+2)}, \ldots, a_{(i+k-3)}, a_{(i+k-1)}, a_n\}, \\
& \{a_i, a_{(i+1)}, a_{(i+2)}, \ldots, a_{(i+k-3)}, a_{(i+k-1)}, a_{(n-1)}\}, \\
& \ldots, \\
& \{a_i, a_{(i+1)}, a_{(i+2)}, \ldots, a_{(i+k-3)}, a_{(i+k-1)}, a_{(i+k)}\}, \\
& \{a_i, a_{(i+1)}, a_{(i+2)}, \ldots, a_{(i+k-3)}, a_{(i+k)}, a_{(i+k+1)}\}, \\
& \{a_i, a_{(i+1)}, a_{(i+2)}, \ldots, a_{(i+k-3)}, a_{(i+k)}, a_{(i+k+2)}\}, \\
& \ldots, \\
& \{a_i, a_{(i+1)}, a_{(i+2)}, \ldots, a_{(i+k-3)}, a_{(i+k)}, a_n\}, \\
& \ldots, \\
& \{a_i, a_{(i+1)}, a_{(i+2)}, \ldots, a_{(i+k-4)}, a_{(i+k)}, a_{(i+k+1)}\}, \\
& \ldots, \\
& \{a_i, a_{(i+2)}, a_{(i+3)}, \ldots, a_{(i+k-1)}, a_{(i+k)}\}, \\
& \{a_i, a_{(i+2)}, a_{(i+3)}, \ldots, a_{(i+k-1)}, a_{(i+k+1)}\}, \\
& \ldots, \\
& \{a_i, a_{(i+2)}, a_{(i+3)}, \ldots, a_{(i+k-1)}, a_n\}, \\
& \ldots, \\
& \{a_i, a_{(n-k+2)}, a_{(n-k+3)}, a_{(n-k+4)}, \ldots, a_n\}\}
\end{aligned}
$$

for $i = 1, 2, \ldots, n - k + 1$.

We can examine the form and claim that two consecutive vertices are adjacent because the intersection of both is $k - 1$. Thus, there is hamiltonian path in every subgraph of the graph $G(n, k)$ of the Johnson Scheme.

Lemma 2: *The first vertex of subgraph S_i is adjacent to the first vertex of subgraph $S_{(i+1)}$.*

Proof:

Let A_1 be the first vertex of subgraph S_1 and B_1 is the first vertex of subgraph S_2. It follows that $A_1 = \{1, 2, 3, \ldots, k\}$, $B1 = \{2, 3, 4, \ldots, k + 1\}$, So, the intersection between A_1 and B_1 is $\{2, 3, 4, \ldots, k\}$ which has cardinality $k - 1$. Thus, they differ in only one element. For A_1, it differs one element from B_1 because it does not have element $k+1$. And B_1 differs one element from A_1 because it does not have element 1. Thus, A_1 is adjacent to B_1. Let C_1 be the first vertex of subgraph S_3, that is $C_1 = \{3, 4, 5, \ldots, k + 2\}$. The intersection between B_1 and C_1 is $\{3, 4, 5, \ldots, k + 1\}$ which has cardinality $k - 1$. So, B_1 is adjacent to C_1. This adjacency also occurs between C_1 and D_1, D_1 and E_1, and so on, for D_1 is the first vertex of subgraph S_4, and E_1 is the first vertex of subgraph S_5. In fact, the first vertex of subgraph S_i is always adjacent to the first vertex of subgraph $S_{(i+1)}$ of the graph $G(n, k)$ of the Johnson Scheme.

Lemma 3: *The last vertex of subgraph S_i is adjacent to the last vertex of subgraph S_j , where $i, j \leq n - k + 1$.*

Proof:

Let A_y be the last vertex of subgraph S_1, By be the last vertex of subgraph S_2 and C_y be the last vertex of subgraph S_3 such that

$$A_y = \{a_1, a_{(n-k+2)}, a_{(n-k+3)}, a_{(n-k+4)}, \ldots, a_n\},$$
$$B_y = \{a_2, a_{(n-k+2)}, a_{(n-k+3)}, a_{(n-k+4)}, \ldots, a_n\}, \text{ and}$$
$$C_y = \{a_3, a_{(n-k+2)}, a_{(n-k+3)}, a_{(n-k+4)}, \ldots, a_n\}.$$

We shall get that the intersection between A_y and B_y is $\{a_{(n-k+2)}, a_{(n-k+3)}, a_{(n-k+4)}, \ldots, a_n\}$, the intersection between A_y and C_y is also $\{a_{(n-k+2)}, a_{(n-k+3)}, a_{(n-k+4)}, \ldots, a_n\}$, and the intersection between B_y and C_y is also $\{a_{(n-k+2)}, a_{(n-k+3)}, a_{(n-k+4)}, \ldots, a_n\}$ which has cardinality $k - 1$. We can also say that among A_y, B_y, and C_y, they differ in only one element. For A_y, it differs one element from B_y because it does not have element a_2, and it differs one element from C_y because it does not have element a_3. For B_y, it differs one element from A_y because it does not have element a_1, and it differs one element from C_y because it does not have element a_3. Thus, A_y is adjacent to B_y and C_y. This adjacency also occurs between B_y and C_y, B_y and D_y, C_y and D_y, and so on, for D_y is the last vertex of subgraph S_4. Thus, the last vertex of subgraph S_i is always adjacent to the last vertex of subgraph S_j of the graph $G(n, k)$ of the Johnson Scheme, where $i, j \leq n - k + 1$.

Lemma 4: *The first vertex of subgraph $S_{(n-k)}$ is adjacent to the vertex of subgraph $S_{(n-k+1)}$.*

Proof:

Note that in the last subgraph, $S_{(n-k+1)}$, has only one vertex. This vertex is the first vertex and also the last vertex of subgraph $S_{(n-k+1)}$. So, this lemma is already proven in Lemma 2.

Theorem: *There is hamiltonian cycle in graph $G(n, k)$ of the Johnson Scheme.*

Proof:

We shall use the above lemmas to prove this theorem. Let S_i is the ith subgraph, for $i = 1, 2, \ldots, n - k + 1$. If i is odd, we make an edge between the first vertex of subgraph S_i and the first vertex of subgraph $S_{(i+1)}$ (Lemma 2), If i is even, we make an edge between the last vertex of subgraph S_i and the last vertex of subgraph $S_{(i+1)}$ (Lemma 3). If i is equal to $n - k + 1$, we make an edge between the last vertex of subgraph S_1 and the vertex of subgraph $S_{(n-k+1)}$. Now, every subgraph has been connected. At last, Lemma 1 has proven that there is hamiltonian path in every subgraph. Thus, there is hamiltonian cycle in the graph $G(n, k)$ of the Johnson Scheme.

Based on the lemmas and theorem, we claim that:

Corollary: *The graph $G(n, k)$ of the Johnson Scheme is hamiltonian.*
Consider this example, $n = 7$ and $k = 3$, then graph $G(7, 3)$ has subgraphs as follows:

S_1 = {{1,2,3}, {1,2,4}, {1,2,5}, {1,2,6}, {1,2,7}, {1,3,7}, {1,3,6}, {1,3,5}, {1,3,4}, {1,4,5}, {1,4,6}, {1,4,7}, {1,5,7}, {1,5,6}, {1,6,7}}
S_2 = {{2,3,4}, {2,3,5}, {2,3,6}, {2,3,7}, {2,4,7}, {2,4,6}, {2,4,5}, {2,5,6}, {2,5,7}, {2,6,7}}
S_3 = {{3,4,5}, {3,4,6}, {3,4,7}, {3,5,7}, {3,5,6}, {3,6,7}}
S_4 = {{4,5,6}, {4,5,7}, {4,6,7}}
S_5 = {{5,6,7}}

If we draw those subgraphs with edges to see the hamiltonian cycle, then we shall get *Figure 2*.

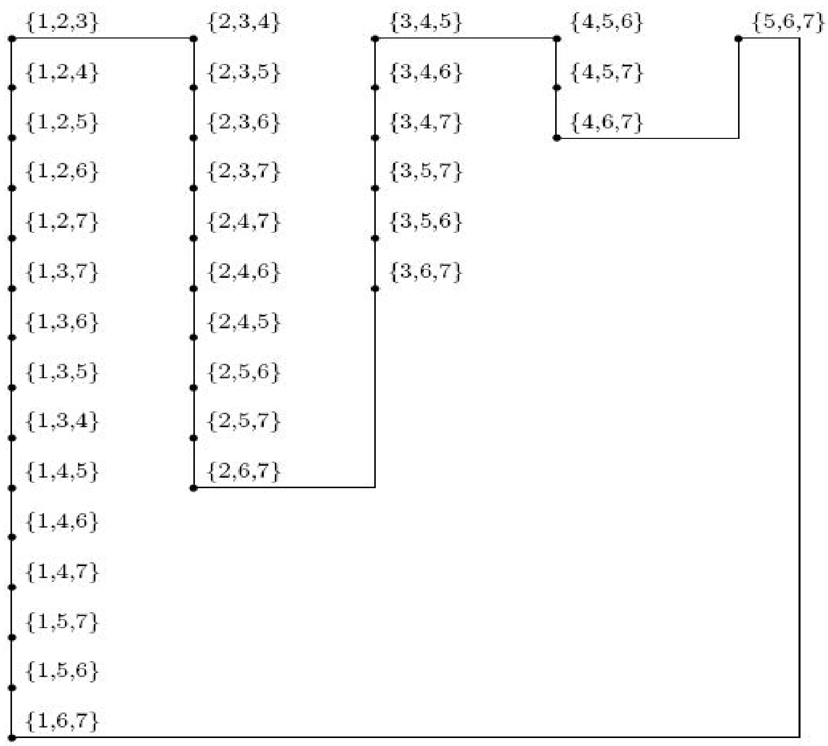

Figure 2 Hamiltonian cycle of the Graph G(7,3) of the Johnson Scheme

4. Conclusion

In this paper, on the graph $G(n, k)$ of the Johnson Scheme, we divide the whole graph into subgraphs which will make us easier to determine one of the graph property.

The whole discussions on the graph $G(n, k)$ of the Johnson Scheme above give conclusion that the graph $G(n, k)$ of the Johnson Scheme is hamiltonian.

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